Multiple states in open channel flow

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Abstract

Steady flow regimes in a free surface flow approaching an obstacle are described and extensively discussed. Attention is focused on the phenomenon of hydraulic hysteresis, and a simple one-dimensional theory to predict its occurrence in a supercritical channel flow is proposed. It is shown that in many cases knowledge of the Froude number of the undisturbed approaching flow and of a geometric characteristic of the obstacle allows for a reliable prediction of the flow state. In the region of multiple regimes, however, the previous history of the flow must also be known.

Three different obstacles in a rectangular channel are considered, namely a sill, a vertical sluice gate, and a circular cylinder, and the theoretical boundaries of the hysteresis region are specified for each obstacle. The experimental results show that the theoretical predictions are consistent with experiments in the case of obstacles that do not affect channel width (*i.e.* sills and gates). On the contrary, in the case of channel contraction, a further parameter, which the presented theory does not account for, was found to affect the behavior of the flow, namely the ratio of undisturbed flow depth to contraction width.

Finally, in the case of a vertical sluice gate it was found that hysteresis develops in a subcritical undisturbed approaching flow as well.

1 Introduction

The occurrence of steady rapidly varied flow in the vicinity of short obstacles is not unusual in open channels. Prediction of flow characteristics as a function of undisturbed approaching flow conditions and obstacle geometry is a primary objective, mainly related to design purposes. In fact, this information must usually be known in order to establish whether or not a constriction is severe enough to influence the upstream flow and to accurately estimate any possible increase in flow depth upstream of the obstacle.

Different kinds of obstacles exist. Any sudden and local change in flow boundaries that obstructs regular flow can be considered an obstacle. Obstruction can be due to variations in bed topography (*e.g.* sills and sharp- or broad-crested weirs), changes in channel width (*e.g.* bridge piers and contractions), or restrictions in flow depth due to the presence of underflow gates. Obstacles that combine two or more of these obstructions may also exist.

Interaction between the flow and the obstacle produces effects that are independent from both the type of obstacle and the undisturbed flow regime, at least from a qualitative point of view. In fact, for both supercritical and subcritical undisturbed approaching flow conditions two general classes of rapidly varied flow can be identified. In the first one, referred to as “weak interaction” (WI), the flow continues undisturbed when passing the obstacle, *i.e.* the flow regime does not change at the obstacle even though flow depth varies locally. Moreover, in this case the specific energy of the flow anywhere along the channel is greater than the minimum specific energy required to pass the obstruction. In the second class, referred to as “strong interaction” (SI), not only do flow depth variations occur in the channel but transition does as well, *i.e.* flow regime changes from supercritical to subcritical and/or *vice versa*. In this case, the specific energy just upstream of the obstacle is equal to the minimum specific energy required to pass the obstacle.

On the contrary, the actual flow configuration in the vicinity of the obstacle depends on the undisturbed flow regime, *i.e.* whether the undisturbed flow approaching the obstacle is supercritical or subcritical. In particular, if a supercritical undisturbed flow approaches the obstacle, the following two different configurations may be established. In a WI class rapidly varied flow, supercritical conditions persist both upstream and downstream of the obstruction and at the obstruction as well. On the contrary, SI causes the upstream flow to undergo a transition from supercritical to subcritical regime and undisturbed conditions are only restored downstream of the obstacle.

The occurrence of both these different steady-state configurations can be predicted as a function of fundamental flow and obstacle parameters, at least within a simple one-dimensional theoretical approach. Nevertheless, a unified theory of rapidly varied flow has not been developed yet. This may be the reason why even specialists only marginally know a somewhat surprising feature that may arise when an undisturbed supercritical flow approaches an obstacle. The theoretical boundaries of steady flow regimes in the vicinity of an obstruction clearly show that a region in the space of fundamental parameters exists where both the above flow states, *i.e.* WI and SI, may occur for the same external steady conditions. The state that is actually established depends on the history of the flow, *i.e.* on the way in which flow conditions have evolved up to the current ones, thus implying the hysteretic character of the flow.

The occurrence of hysteresis in a supercritical flow approaching a sill has been recognized and widely investigated both experimentally and theoretically over the past few decades [1–8]. On the contrary, much less effort has been devoted to the study of hydraulic hysteresis for other types of obstacles, such as channel constriction. Nevertheless, some experimental studies reported in the literature qualitatively

have confirmed the hysteretic behavior of a supercritical flow through channel constriction [9, 10]. More recently, the existence of hysteresis in flow under a sluice gate has been demonstrated both theoretically and experimentally [11].

Here, our primary objectives are to examine hydraulic hysteresis in general terms, *i.e.* independently of the type of obstacle, and review significant results concerning specific obstructions. A theoretical approach is presented which uses simple relationships expressing energy balance and momentum conservation to infer a criterion able to predict the conditions in which hysteresis occurs in a supercritical flow. This general criterion is then applied to some specific obstacles, namely a sill, a vertical sluice gate, and a circular pier. The theoretical predictions are compared with available experimental data. Finally, some conclusions concerning the main features of the investigated phenomenon are reported, and situations where the proposed theoretical approach fails are highlighted.

2 One-dimensional approach

A simple criterion to identify hysteresis occurrence in a rapidly varied flow has been recently proposed by Defina and Susin [11]. Here, we shortly recall the fundamental aspects of this theoretical approach and show that the existence of hysteresis emerges when sufficient and necessary conditions for WI and SI interaction occurrence are both considered.

We consider a one-dimensional supercritical undisturbed channel flow of velocity v_0 and depth y_0 approaching a generic obstacle. The frictional effects of the boundaries are considered and the pressure is assumed to be hydrostatic except in the proximity of the obstacle. A non-horizontal channel is considered. The bottom slope is assumed to be small enough that $\cos(\vartheta) \simeq 1$, $\sin(\vartheta) \simeq \tan(\vartheta)$, ϑ being the angle of the channel bed to the horizontal.

To describe the phenomenon of hysteresis, it is convenient to use the concept of specific energy E , *i.e.* mechanical energy per unit weight of flow relative to the bottom of the channel. Thus, for the *undisturbed approaching flow*

$$E_0 = y_0 + \frac{v_0^2}{2g} \quad (1)$$

where g is gravity. Moreover, the minimum specific energy required to pass the obstacle is denoted with E_{min} .

It is well known that if E_0 is lower than E_{min} , then the approaching flow does not have enough energy to pass the obstacle and, as a consequence, it necessarily undergoes a transition from supercritical to subcritical regime. A stationary hydraulic jump occurs far from the obstacle and a subcritical backwater profile is established after the jump. The backwater profile, along which E increases, allows for the minimum specific energy to be achieved at the section just before the obstacle. In this situation, a “strong interaction” between the flow and the obstacle occurs. Hence, the sufficient condition for SI (*i.e.* supercritical-subcritical transition in the upstream reach of the channel, fig. 1b) to occur is

Figure 1: Supercritical channel flow approaching an obstacle. E_0 denotes the specific energy of the *undisturbed supercritical flow*. E_{min} denotes the minimum specific energy required to pass the obstacle. Case a represents “weak interaction” conditions. Cases b and c represent “strong interaction” conditions with $E_0 < E_{min}$ and $E_0 > E_{min}$, respectively.

$$E_0 < E_{min} \quad (2)$$

The necessary condition for SI to occur can be inferred by the use of the total energy conservation equation between the flow just upstream of the obstacle and the flow at the obstacle [11]. As long as a hydraulic jump is established in the upstream reach of the channel (fig. 1b and c), we have

$$E_0 - \Delta H_J \leq E_{min} \quad (3)$$

where ΔH_J denotes the energy loss at the hydraulic jump.

It is now worth noting that conditions complementary to eqns. (2) and (3) give the necessary and sufficient conditions for WI to occur, respectively (Figure 1a). In fact, if supercritical conditions are maintained at the obstacle, then the specific energy of the *undisturbed approaching flow* must be greater than E_{min} , *i.e.*

$$E_0 \geq E_{min} \quad (4)$$

Moreover, if the following condition holds

$$E_0 - \Delta H_J > E_{min} \quad (5)$$

then the specific energy of the flow just upstream of the obstacle is certainly greater than the minimum specific energy required to pass the obstacle, *i.e.* the flow is supercritical everywhere.

On the basis of the above considerations, we can now state that if the specific energy of the *undisturbed supercritical approaching flow*, E_0 , satisfies the conditions expressed by eqns. (3) and (4), then both WI and SI flow regimes can exist. Moreover, for a given undisturbed flow condition and given geometric characteristics of the obstacle, E_{min} may depend on the flow regime being established just upstream of the obstacle. Hence, if the minimum specific energy for the supercritical (WI) and subcritical (SI) regimes just upstream of the obstacle are denoted with E_l and E_L , respectively, both steady flow configurations, *i.e.* with or without upstream transition, are possible and stable as long as the following constraint is met

$$E_l \leq E_0 \leq E_L + \Delta H_J \quad (6)$$

In this situation, the “history” of the flow plays a crucial role, as it determines the state that is actually established in the vicinity of the obstruction [5, 8]. For this reason, the behavior of the flow is said to be “hysteretic”.

In the following sections, three different types of obstacles are examined. Condition eqn. (6) is specified for each obstacle, and the amplitude of the hysteresis domain is expressed in terms of the fundamental flow parameters and geometrical characteristics of the obstruction.

3 Flow over a sill

We will now develop condition eqn. (6) for the case of flow over a sill in a rectangular channel in terms of the Froude number of the undisturbed approaching flow, F_0 , and the non-dimensional step height $\Delta z/y_c$, where y_c is the critical depth for a given flow rate and channel width. In this case (fig. 2), the threshold values for specific energy are

$$E_l = \Delta z + E_c + \Delta H_l \quad E_L = \Delta z + E_c + \Delta H_L \quad (7)$$

Figure 2: Steady flow regimes for a supercritical undisturbed flow approaching a sill, with notation. Limit conditions for the weak interaction regime (left) and strong interaction regime (right) are shown.

where Δz is the height of the sill, E_c is the critical specific energy, and ΔH_I and ΔH_L are the energy losses just before the section at which the critical condition for WI and SI regimes, respectively, is attained.

Combining eqns. (6) and (7), and recalling that

$$E_c/y_c = 3/2, \quad E_0/y_c = F_0^{-2/3} + F_0^{4/3}/2, \quad \text{and} \quad (8)$$

$$\Delta H_j/y_c = \frac{F_0^{-2/3}}{16} \frac{(\sqrt{1+8F_0^2}-3)^3}{(\sqrt{1+8F_0^2}-1)}$$

the following expression can be easily found for the lower and upper boundaries of the hysteresis region

$$\frac{\Delta z_l}{y_c} = F_0^{-2/3} + F_0^{4/3}/2 - 3/2 - \Delta H_I/y_c \quad (9)$$

$$\frac{\Delta z_L}{y_c} = F_0^{-2/3} + \frac{F_0^{4/3}}{2} - \frac{3}{2} - \frac{F_0^{-2/3}}{16} \frac{(\sqrt{1+8F_0^2}-3)^3}{(\sqrt{1+8F_0^2}-1)} - \Delta H_L/y_c \quad (10)$$

The above result has already been proposed in previous studies [3–6] which neglected energy losses ΔH_I and ΔH_L . In eqns. (9) and (10), the nondimensional energy losses $\Delta H_I/y_c$ and $\Delta H_L/y_c$ both depend on F_0 and the geometry of the step. Although these losses can be correctly evaluated only through experiments, a rough estimation based on mass and momentum conservation equations is possible, and given here below. Let us first consider the case of the WI regime. Using the notation adopted in fig. 3a and the concept of momentum function M (*i.e.* the sum of pressure force and momentum flux, per unit width and unit weight of fluid, at a given section of the flow [12]), the equilibrium condition in the horizontal direction is

$$M_u - M_d - m_S = 0 \quad (11)$$

where M_u and M_d denote the upstream and downstream values of M , respectively, and m_S is the force, in the flow direction, exerted by the upward face of the step (per unit width and unit weight of fluid). Assuming that at the section immediately

Figure 3: Control volume (CV) selected for the application of the momentum balance equation in the case of weak interaction regime (a) and strong interaction regime (b).

upstream of the step the flow depth is $y_c + \Delta z_l$ and the pressure is hydrostatic [3], then m_s is given as

$$m_s \cong (y_c + \Delta z_l/2)\Delta z_l \quad (12)$$

Noting that when the WI $\rightarrow$ SI limit condition occurs $M_u = M_0$ and $M_d = M_c$, and recalling that

$$\frac{M_0}{y_c^2} = \frac{F_0^{-4/3}}{2} + F_0^{2/3} \quad \frac{M_c}{y_c^2} = \frac{3}{2} \quad (13)$$

eqn. (11) can be rearranged to read

$$\left(\frac{\Delta z_l}{y_c} + 1 \right)^2 = 2F_0^{2/3} + F_0^{-4/3} - 2 \quad (14)$$

Finally, combining eqns. (9) and (14) gives

$$\frac{\Delta H_l}{y_c} = F_0^{-2/3} + \frac{F_0^{4/3}}{2} - \frac{1}{2} - \sqrt{F_0^{-4/3} - F_0^{2/3} - 2} \quad (15)$$

The above procedure is also used to compute energy loss for the SI regime (fig. 3b). In this case, the horizontal force exerted by the upward face of the step, m_s is

$$m_s \cong (y_U - \Delta z_L/2)\Delta z_L \quad (16)$$

where the water depth just upstream of the obstacle, y_U , is the conjugate of the undisturbed flow depth, *i.e.*

$$\frac{y_U}{y_c} = \frac{F_0^{-2/3}}{2} \left(-1 + \sqrt{1 + 8F_0^2} \right) \quad (17)$$

Figure 4: Plot of limit conditions for both weak and strong interaction in the plane $(\Delta z/y_c, F_0)$, when energy losses are neglected (solid lines) and included (dashed lines).

The equilibrium condition expressed by eqn. (11) gives

$$\frac{\Delta z_L}{y_c} = \frac{F_0^{-2/3}}{2} \left(\sqrt{1 + 8F_0^2} - 1 \right) - \sqrt{3 - F_0^{4/3} \left(\sqrt{1 + 8F_0^2} - 1 \right) / 2} \quad (18)$$

and combining eqns. (18) and (10) then gives

$$\frac{\Delta H_L}{y_c} = \frac{F_0^{-2/3} - 6}{4} + \frac{F_0^{-2/3}}{\sqrt{1 + 8F_0^2} - 1} + \sqrt{3 - \frac{F_0^{4/3}}{2} \left(\sqrt{1 + 8F_0^2} - 1 \right)} \quad (19)$$

The plot of eqns. (9) and (10) is shown in fig. 4. Solid lines are used when energy losses are neglected, dashed lines when energy losses are expressed by eqns. (15) and (19). In both cases, three different regions can be distinguished in the plane $(\Delta z/y_c, F_0)$. The first region, extending above the curve $\Delta z_L/y_c$, corresponds to WI conditions with a supercritical flow extending along the whole channel. In the second region, below the curve $\Delta z_l/y_c$, the flow necessarily undergoes a supercritical to subcritical transition upstream of the step (*i.e.* SI). Finally, the hysteresis region, in which both stable states are possible, lies between these two curves.

Figure 5: Steady flow profiles for a supercritical flow over a step, with notation.

It is worth noting that when energy losses are neglected, the hysteresis region is considerably wide (fig. 4). On the contrary, when energy losses are considered, the WI→SI limit condition moves towards the SI→WI limit condition and the hysteresis region is much smaller. Surprisingly, most fluid mechanics textbooks propose eqn. (9) with $\Delta H_l = 0$ as the limit condition for WI↔SI.

Before we go any further, it is important to emphasize that the assumptions introduced in order to obtain eqns. (15) and (19) allow for a reliable prediction of the qualitative behavior of the hysteresis boundaries, but do not always apply for a quantitative analysis. In particular, m_s is reasonably approximated as previously reported only when the flow just upstream of the obstacle is subcritical (*i.e.* for the SI configuration). Indeed, in this case losses are rather small and do not significantly affect the inviscid solution. On the contrary, when the flow just upstream of the obstacle is supercritical, energy losses strongly depend on the shape of the step and on the ratio y_0/S , where S is the length of the upstream face of the step (fig. 5). Equation (12) for m_s allows for a reliable evaluation of energy losses provided that the ratio y_0/S is not much smaller than one (fig. 5, profile A). Otherwise, *i.e.* when $y_0 \ll S$ (fig. 5, profile C), eqn. (12) strongly overestimates m_s and thus the amount of energy lost by the current in passing the step as well.

It is also worth highlighting that the present theory assumes hydrostatic pressure distribution. On the contrary, at both SI→WI and WI→SI limit conditions free surface slope and curvature across the step are not negligibly small. However, their effects can be accounted for by introducing a suitable *equivalent energy loss or gain* as suggested by Marchi [13], to be included in the terms ΔH_l and ΔH_L .

By analyzing the experimental data available in the literature and comparing them with the present theoretical results we can provide an adequate explanation of the above statements.

Figure 6 compares our theoretical results with the experimental data of Muskatirovic and Batinic [3]. It can be observed that when the step is not severe, *i.e.* $S = 2\Delta z$ (fig. 6a), experimental points collapse onto the theoretical curves which include energy losses, thus confirming the reliability of the assumptions made to compute ΔH_l and ΔH_L . When $S = 0$ (fig. 6b), Muskatirovic and Batinic [3] do not observe any hysteresis. The experimental points at the WI→SI and SI→WI limit conditions collapse onto a single curve which approximately corresponds to the theoretical upper limit, $\Delta z_L/y_c$, implying that eqn. (19) (weakly) underestimates losses.

Figure 6: Experimental data of Muskatirovic and Batinic [3]. (a) Full symbols denote the WI→SI limit condition, open symbols denote the SI→WI limit condition; (b) symbols denote the WI↔SI limit condition. The curves are the same as those in fig. 4.

An interesting set of experimental data is provided by Baines and Whitehead [8]. They measured the limits for SI→WI and WI→SI in a short upward sloping channel. This device somewhat resembles a step with a gently sloping upstream face (*i.e.* with large S) and $T=0$ (fig. 7). When the flow is supercritical along the channel (fig. 7a), then the free surface profile qualitatively corresponds to profile C of fig. 5 and no energy losses should be included. Nevertheless, in this case, the upstream face of the step is rather long. Consequently, bottom friction is not entirely negligible and a small energy loss has to be incorporated in $\Delta H_l/y_c$.

Figure 8 compares experimental findings by Baines and Whitehead [8] with present theoretical curves. The experimental points describing the WI→SI limit are arranged just below the theoretical curve with $\Delta H_l/y_c=0$, in agreement with the above discussion about energy losses affecting the flow in this case. On the contrary, the experimental behavior of the SI→WI limit is somewhat surprising. The measured points are arranged above the theoretical curve, implying that the flow energy increases along the channel. A close inspection of the free surface profile when the flow is subcritical along the whole channel and the hydraulic jump is attached to the downstream face of the rounded sluice gate (fig. 7b), shows that near the downstream end of the channel both the free surface slope and curvature

Figure 7: Experimental device adopted by Baines and Whitehead [8] to study hydraulic hysteresis. Flow along the short upward sloping channel is supercritical (a) and subcritical (b).

are negative and large. As previously discussed, their effects can be accounted for by introducing a suitable equivalent energy gain to be included in $\Delta H_L/y_c$.

The following conclusions can be drawn from the above discussion. First of all, we can see that the theory presented here gives reliable predictions of hysteresis limits, provided that energy losses ΔH_l and ΔH_L are properly evaluated. Moreover, it is worth pointing out that most fluid mechanics textbooks propose eqn. (9) with $\Delta H_l = 0$ as the WI $\leftrightarrow$ SI limit condition. This disagrees with the present results and experimental evidence, which suggest that, in most cases, the inviscid solution given by eqn. (10) is a more suitable approximation for the WI $\rightarrow$ SI limit.

Figure 8: Present theoretical limits of the hysteresis region compared to the experimental data of Baines and Whitehead [8]. Full symbols denote the WI $\rightarrow$ SI limit, open symbols denote the SI $\rightarrow$ WI limit. The curves are the same as those in fig. 4.

Figure 9: Sketch of possible steady flow configurations when an undisturbed supercritical flow approaches a vertical sluice gate. (a) Undisturbed flow conditions along the channel (WI); (b) free outflow conditions at the gate with an upstream transition from supercritical to subcritical flow (SI).

4 Flow under a sluice gate

The hysteretic behavior of a flow under a vertical sluice gate was thoroughly described both theoretically and experimentally by Defina and Susin [11] and, to our knowledge, this is the only investigation on this topic. Here, we will briefly recall the theoretical aspects, experimental apparatus and procedure. We will focus particular attention on comparing theoretical predictions and experimental results.

Figure 9 shows possible steady flow configurations when an undisturbed supercritical flow approaches a vertical sluice gate. In the WI case (fig. 9a), the height of the gate opening is sufficiently larger than the undisturbed water depth so that smooth undisturbed flow conditions exist along the channel. In the SI case (fig. 9b), transition from supercritical to subcritical regime takes place upstream of the gate and, at the obstacle, the free outflow condition is established (*i.e.* the fluid issues from under the gate as a jet of supercritical flow with a free surface open to the atmosphere). The hysteretic behavior of the flow concerns the possibility that, for a given F_0 , both SI and WI configurations may exist for the same opening of the gate.

In both the above situations, energy losses at the gate can usually be neglected. Hence, the threshold specific energies E_L and E_l , as well as the related openings of the gate a_L and a_l , can be derived by using the Bernoulli equation between the upstream and the downstream flows (the latter being the flow at the *vena contracta*, in the SI case). In the case of a rectangular channel, the flow is shown to be hysteretic as long as the nondimensional opening of the gate, a/y_c is such that

$$a_l/y_c \leq a/y_c \leq a_L/y_c \quad (20)$$

where a_l/y_c and a_L/y_c are related to the Froude number of the undisturbed approaching flow F_0 as expressed by the following equations

$$a_l/y_c = F_0^{-2/3} \quad (21)$$

$$\left(\frac{a_L}{y_c} c_c\right) + \frac{1}{2} \left(\frac{a_L}{y_c} c_c\right)^{-2} = \frac{4F_0^{4/3}}{1 + \sqrt{1 + 8F_0^2}} + \frac{2F_0^{4/3}}{(-1 + \sqrt{1 + 8F_0^2})^2} \quad (22)$$

In eqn. (22) c_c is the contraction coefficient (*i.e.* the ratio of the flow depth at the *vena contracta* to the height of the gate opening). A reliable evaluation of a_L/y_c requires a reliable evaluation of c_c as well. The following approximate equation will from here on be adopted

$$c_c = 1 - r(\zeta) \sin(\zeta) \quad (23)$$

$$a_L/y_U = 1 - r(\zeta)[1 - \cos(\zeta)] \quad (24)$$

$$r(\zeta) = 0.153\zeta^2 - 0.451\zeta + 0.727 \quad (25)$$

where ζ is a dummy parameter and y_U is the flow depth immediately upstream of the gate. A detailed discussion about the evaluation of c_c and eqns. (21) to (25) is reported in ref. [11].

Figure 10 shows the behavior of a_l/y_c and a_L/y_c in the plane $(a/y_c, F_0)$. Possible flow regimes in regions bounded by these two curves are also specified. It is worth pointing out that the ascending branch AM of the curve a_L/y_c is replaced with the horizontal line $A'M$: $a/y_c = (a/y_c)_{max} \cong 1.15$ (see Appendix A of Defina and Susin [11] for details). Hence, the boundaries of the hysteresis region do not merge at $F_0 = 1$. This suggests that the dual solution domain in the $(a/y_c, F_0)$ plane may close in the region of subcritical undisturbed approaching flows (*i.e.* $F_0 < 1$). Indeed that is exactly what is found to happen once the theoretical analysis of the flow under a sluice gate is extended to subcritical undisturbed flow condi-

Figure 10: Steady flow regimes in the vicinity of a vertical sluice gate for a super-critical undisturbed approaching flow on the $F_0 - a/y_c$ diagram.

Figure 11: Sketch of possible steady flow configurations when an undisturbed subcritical flow approaches a vertical sluice gate. (a) undisturbed flow conditions along the channel (WI); (b) free outflow conditions at the gate with downstream transition from supercritical to subcritical flow (SI).

tions. In this case, three different steady states may be established in the vicinity of the gate: undisturbed flow conditions when WI occurs, and either free or submerged outflow in the case of SI.

The flow continues undisturbed as long as the gate does not touch the free surface (fig. 11a). Therefore, when WI conditions are established in the channel, the non-dimensional gate opening is certainly greater than a_l/y_c , where a_l/y_c depends on F_0 as given by eqn. (21). Once the gate has touched the free surface, either submerged or free outflow is established, depending on the opening of the gate. The limit value of the gate opening (denoted with a_L/y_c) between these two configurations is such that the related flow depth at the *vena contracta* equals the conjugate depth of the downstream undisturbed subcritical flow (fig. 11b). Hence, the momentum balance between the *vena contracta* and the downstream flow gives

$$\left(\frac{a_L}{y_c} c_c\right) = \frac{F_0^{-2/3}}{2} \left(-1 + \sqrt{1 + 8F_0^2}\right) \quad (26)$$

Curves a_l/y_c and a_L/y_c for $F_0 = 1$ are plotted in fig. 12. The branch AM' of the curve a_L/y_c is replaced with the horizontal line $A'M'$: $a_L/y_c = (a_L/y_c)_{max} \cong 1.15$ as was the case for supercritical approaching flow. As a consequence, the region $AA'B$ exists, which lies below the boundary a_L/y_c and above the boundary a_l/y_c . In other words, the region $AA'B$ is the extension of the hysteresis region into the subcritical domain. In this region, both undisturbed and free outflow steady states are possible for given a/y_c and F_0 . It is worth noting that the ranges of both a/y_c and F_0 in which hysteresis is found to occur in the subcritical domain are rather wide ($1 \leq a/y_c < 1.15$, $0.8 < F_0 \leq 1$). Therefore, hysteresis can be expected to manifest itself in many practical cases.

An extensive series of experimental results [11] can be used to compare experimental and theoretical boundaries of the hysteresis region. We will first give a brief description of the adopted experimental procedure. Possible flow regimes were investigated for different values of the gate opening while maintaining a fixed flow rate (*i.e.* a fixed undisturbed Froude number F_0). Quasi-uniform flow conditions were initially established in the flume with the gate opening larger than the

Figure 12: Steady flow regimes in the vicinity of a vertical sluice gate for a subcritical undisturbed approaching flow on the $F_0 - a/y_c$ diagram.

undisturbed flow depth. In order to examine the hysteresis phenomenon, the gate was first lowered until it touched the free-surface, so that SI conditions suddenly occurred, and then gradually raised at small steps. The gradual lifting of the gate was protracted until the WI flow configuration was suddenly restored in the channel.

The experimental undisturbed Froude number ranged from about 0.72 to about 4.57. In all the experiments, the WI→SI limit condition occurred as soon as the gate touched or slightly passed the free-surface. In particular, for $F_0 > 0.8$ free outflow conditions were established, while for $F_0 < 0.8$ the outflow was submerged. This occurrence not only confirms the behavior of the limit a_L/y_c theoretically predicted (actually this result was somewhat obvious) but it also substantiates the physical meaning of the branch $A'M'$ in fig. 12. $F_0 \cong 0.8$ was also found to be the smallest undisturbed Froude number for which hysteresis occurred. In fact, for $F_0 < 0.8$, undisturbed conditions were found to be restored as soon as the gate was raised just above the undisturbed flow depth. On the contrary, for $F_0 > 0.8$, the SI→WI limit condition only occurred when the opening of the gate was much larger than the undisturbed flow depth.

The hysteretic character of the flow is clearly shown in fig. 13, where the behavior of the non-dimensional flow depth just upstream of the gate, y_U/y_c is described as a function of the gate opening a/y_c . Experimental conditions for $F_0 = 0.98$ are plotted. The existence of two different steady configurations in the vicinity of the gate for the same a/y_c but different previous states of the flow is evident.

Figure 14 gives a comprehensive plot of all the experimental conditions at which flow reversion from SI to WI suddenly occurred. The theoretical boundaries of the hysteresis region are plotted as well. The experimental points $(F_0, a_L/y_c)$ agree well with the upper theoretical boundary a_L/y_c both qualitatively and quantitatively. In particular, they clearly exhibit the horizontal trend in the range $0.81 < F_0 < 1.65$.

It is worth pointing out that, as previously observed, the reliability of the theoretical limit a_L/y_c is strictly related to the equation adopted to evaluate the contraction coefficient c_c . In other words, the more physically based c_c is, the more reliable

Figure 13: Experimental hysteresis loop in the $a/y_c - y_U/y_c$ diagram. The Froude number of the undisturbed approaching flow is $F_0 = 0.98$. The sequence of experimental points goes clockwise from I (initial undisturbed flow) to R (restored undisturbed flow). The circles and diamonds denote free outflow and undisturbed flow, respectively; the full and open symbols denote the lowering stage and raising stage, respectively.

a_L/y_c is. For example, if the theoretical expression given by Cisotti and Von Mises (as reported by Gentilini [14]) is adopted for c_c , then a_L/y_c behaves as shown in fig. 15 for $F_0 \geq 1$. In this case, gravity effects on the issuing flow are neglected, and both c_c , and a_L/y_c are thus rather poorly estimated. As a consequence, only a qualitative agreement between theoretical predictions and experimental results is found, as shown in fig. 15.

Finally, we can state that the simple theoretical approach outlined in Section 1 also applies fairly well to the case of flow under a vertical sluice gate. In this case, an

Figure 14: Comparison between the theoretical (—) and experimental (o) upper boundary of the hysteresis region. The lower boundary is also plotted for completeness.

Figure 15: Theoretical upper boundary of the hysteresis region (a_L/y_c) when c_c is computed according to Cisotti and Von Mises formula. Experimental data are also plotted for comparison. Symbols are the same as in fig. 14.

accurate evaluation of the contraction experienced by the flow issuing from under the gate is required in order to obtain reliable predictions of the upper hysteresis limit.

5 Flow past a vertical circular cylinder

The theoretical evaluation of minimum specific energies E_I and E_L for the case of channel constriction is far from being an easy task. In this case, in fact, the flow in the vicinity of the obstacle assumes a two-dimensional character (in plan). In addition, free surface slope and curvature, *i.e.* non-hydrostatic pressure distribution along the vertical, strongly affect the flow.

Therefore, we can expect minimum specific energy as determined according to the one-dimensional approach to poorly match experimental conditions. The one-dimensional approach uses only one geometric parameter to describe the obstruction, namely the ratio of obstructed channel width b to the full channel width B (fig. 16c). The shape, length, and position of the obstacle with respect to channel axis and flow depth are neglected. However, these geometric characteristics are very important in determining the limit conditions for both WI and SI configurations, as will be shown in the present section.

Figure 16: Plan view showing some examples of channel constrictions characterized by the same value of the ratio b/B .

In spite of its limitations, the one-dimensional approach can nonetheless provide an initial, even if rough, approximation of limit conditions that can then be compared with true, experimental values. We will now examine the case of a flow around a vertical circular cylinder placed along the axis of a rectangular channel (fig. 16b), both theoretically and experimentally. In this case, the fundamental non-dimensional parameters adopted to develop condition eqn. (6) are the Froude number of the undisturbed approaching flow, F_0 and the ratio b/B , where $b = B - D$, D being the cylinder diameter.

Due to the smooth shape of the obstacle, energy losses at the obstacle are assumed to be negligible. Hence, both E_l and E_L coincide with the critical specific energy at the section of maximum contraction, E_{cr} . Recalling that

$$\frac{E_{cr}}{y_c} = \frac{3}{2}(b/B)^{-2/3} \quad (27)$$

and combining eqns. (6) and (27), the theoretical boundaries of the hysteresis domain can be easily found

$$(b/B)_t = \sqrt{\frac{27}{8}} F_0 \left(1 + \frac{F_0^2}{2}\right)^{-3/2} \quad (28)$$

$$(b/B)_L = \sqrt{\frac{27}{8}} F_0 \left(1 + \frac{F_0^2}{2} - \frac{\left(\sqrt{1 + 8F_0^2} - 3\right)^3}{16\left(\sqrt{1 + 8F_0^2} - 1\right)}\right)^{-3/2} \quad (29)$$

Equations (28) and (29) are plotted in fig. 17. As was the case for the obstacles examined in the previous sections, three different regions can be distinguished in the plane of fundamental parameters. The first region, extending above the curve $(b/B)_L$, corresponds to the WI configuration, with a supercritical flow extending along the whole channel. In the second region, below the curve $(b/B)_t$, the flow necessarily undergoes a supercritical to subcritical transition upstream of the obstacle (*i.e.* SI). Finally, the hysteresis region, in which both stable states are possible, lies between these two curves.

The effects of the physical processes not included in the above theory were examined experimentally. The apparatus is sketched in fig. 18. The experiments were performed in a 0.38m wide, 0.5m high, and 20m long tilting flume with Plexiglas walls, whose bottom slope could be adjusted to a maximum of 5%. Water was supplied by a constant head tank which maintained very steady flow conditions.

A vertical, sharp crested sluice gate was placed at the flume entrance, and a gear system was used to raise and lower the gate. It was possible to set the height of the gate opening with an accuracy of ± 0.2 mm. A vertical cylinder with a diameter in the range $0.060\text{m} < D < 0.205\text{m}$ was placed at the test section, which was located approximately 3m downstream of the channel inlet. One ultra-sonic transducer, movable along the channel axis upstream of the cylinder, measured flow depths with an accuracy of ± 0.5 mm. Water depth was accurately measured at three

Figure 17: Plot of the limit conditions for both weak and strong interaction in the plane $(b/B, F_0)$.

positions, namely at 0.3m, 1.0m, and 2.0m upstream of the cylinder. This made it possible to extrapolate the undisturbed water depth at the test section, y_0 , with an accuracy of ± 1.0 mm.

Each run was conducted according to the following procedure, while maintaining a fixed flow rate in the range $0.01\text{m}^3/\text{s} < Q < 0.06\text{m}^3/\text{s}$. Initially, the height of the sluice gate opening was small enough to ensure supercritical flow conditions from the gate to the end of the flume (*i.e.* WI). Then, the gate opening was increased by small increments so that, at each step, a new steady flow configuration with slightly increased y_0 (*i.e.* slightly decreased F_0) was established in the channel.

Figure 18: Sketch of the experimental apparatus.

The gate raising was protracted until a stationary hydraulic jump was established upstream of the cylinder, *i.e.* until SI conditions occurred. The experimental value of the undisturbed Froude number at the WI→SI limit condition was estimated to be the average of the two undisturbed Froude numbers measured just before and after the jump formed. This limit corresponds to the curve $(b/B)_l$ in fig. 17.

Once the SI configuration was established, the gate opening height was decreased by small steps until the jump was swept away and vanished, *i.e.* WI conditions were restored. The experimental value of the undisturbed Froude number at the SI→WI limit condition was estimated to be the average of the two undisturbed Froude numbers measured just before and after the jump vanished. This limit corresponds to the curve $(b/B)_L$ in fig. 17.

Before discussing the present experimental results, we will give a qualitative description of the flow pattern observed in the vicinity of the obstacle during the experiments.

At the beginning of each run, the flow was supercritical along the whole channel apart from a small area behind the bow shock wave produced by the cylinder (fig. 19a). In these conditions, the detached shock front experienced a regular reflection at the wall. Moreover, the fluid behind the detached shock was driven toward the channel axis by the expansion fan emanating from the cylinder. This produced a nearly straight secondary shock front behind the cylinder which intersected the primary reflected shock.

At Froude numbers lower than the initial F_0 , the reflection of the detached shock wave at the wall became irregular. A Mach stem was established, which was approximately normal to the wall (fig. 19b) although somewhat distorted due to wall friction. Nevertheless, it was clearly recognizable. On the contrary, the slipstream, if there was any, could not be identified. This was possibly related to free surface curvature effects, which diffused the shock wave fronts thus obscuring the pattern of the shock waves.

Further decreases in the Froude number resulted in a progressive increase in the Mach stem height until a hydraulic jump was established upstream of the cylinder (fig. 19c).

The rather complex flow pattern observed in the vicinity of the obstacle at the WI→SI limit condition suggests that the one-dimensional approach might be inadequate to infer minimum specific energy E_l in terms of only two fundamental parameters (*i.e.* b/B and F_0). Actually, the behavior of the boundaries of the hysteresis region is controlled by at least two competing factors: (i) effects of free surface slope and curvature, which are measured by the parameter y_0/b , and (ii) energy losses and two-dimensional effects, which are difficult to discern and are controlled by the ratio b/B . The effects of both y_0/b and b/B will now be analyzed.

A comprehensive plot of all the experimental conditions at which the flow configuration changed from WI to SI and *vice versa* is given in fig. 20. Each plot in fig. 20 refers to a given cylinder diameter (*i.e.* to a given ratio b/B) and contains experimental boundaries of the hysteresis region in the $(y_0/b, F_0)$ plane. Theoretical undisturbed Froude numbers F_{0l} and F_{0L} provided by eqns. (28) and (29) are shown in fig. 20 as well.

Figure 19: Sketch of the front pattern developing in the vicinity of the cylinder.

For the case $b/B = 0.74$, the cylinder diameter was $D = 0.1\text{m}$. However, in order to obtain experimental values of y_0/b up to 0.6, a pair of cylinders of diameter $D = 0.05\text{m}$ aligned normal to the flow direction was also used. Points with $y_0/b > 0.27$ refer to this experimental configuration.

Figure 20: Experimental and theoretical boundaries of the hysteresis region in the $(y_0/b, F_0)$ plane for different values of the ratio b/B . Full symbols denote the WI→SI limit, open symbols denote the SI→WI limit. Solid vertical lines denote one-dimensional theoretical boundaries.

It can be observed that the one-dimensional approach gives the correct order of magnitude of the critical Froude numbers at the SI→WI limit conditions. This result is somewhat surprising if we recall the complex experimental flow pattern previously discussed and shown in fig. 19. Anyway, the dependence of both the upper and lower boundaries of the hysteresis region on the free surface slope and curvature is rather evident. At small values of y_0/b , *i.e.* when energy losses prevail over free surface slope and curvature effects, experimental values of the undisturbed Froude number at the SI→WI limit conditions are greater than those predicted by the inviscid one-dimensional model, as expected. At greater values of y_0/b , effects due to free surface slope and curvature prevail. As observed in Section 3, these effects act as an equivalent gain of energy. Actually, both experimental boundary curves shift towards the left side of the $(y_0/b, F_0)$ plane as y_0/b increases.

Figure 21: Experimental (symbols) and theoretical (solid lines) boundaries of the hysteresis region in the $(b/B, F_0)$ plane. $y_0/b = 0.1$ (left), $y_0/b = 0.15$ (center), and $y_0/b = 0.3$ (right).

The above discussion is confirmed in fig. 21, which shows the behavior of the boundaries of the hysteresis region in the standard $(b/B, F_0)$ plane for three different values of y_0/b .

Figure 22 shows the behavior of experimental minimum specific energies E_l and E_L , normalized with the one-dimensional theoretical values $E_{l_{1D}}$ and $E_{L_{1D}}$, respectively, as a function of the ratio y_0/b .

All the points show a similar trend driven by free surface slope and curvature effects: energy ratios $E_l/E_{l_{1D}}$ and $E_L/E_{L_{1D}}$ decrease with y_0/b increasing, at least in the range $0 < y_0/b < 0.3$. However, experimental data for $b/B = 0.74$ suggest that the above ratios may approach a constant value as y_0/b is further increased.

Finally, it is worth recalling that when $y_0/b \rightarrow 0$, the effects related to the two-dimensional character of the flow are very important in establishing the lower limit E_l . Indeed, fig. 22 shows that the differences between the experimental results and one-dimensional predictions increase with b/B decreasing. Moreover, these differences only slightly decrease with y_0/b increasing, thus suggesting that the two-dimensional character of the flow affects the one-dimensional solution even at moderately high water depths.

On the contrary, when considering the upper boundary of the hysteresis region, experimental E_L slightly differs from $E_{L_{1D}}$, and only a weak dependence of $E_L/E_{L_{1D}}$ on b/B can be recognized.

In this section the occurrence of hysteresis for the case of channel constriction was discussed. In particular, the results of an in-depth theoretical and experimental study of the case of a flow around a vertical circular cylinder were presented. Although this type of obstacle is very simple, as it is characterized by just one length scale, *i.e.* the diameter D , the experimental results showed that the threshold specific energies E_l and E_L (*i.e.* the hysteresis boundaries) are dependent on parameters F_0 , b/B , and y_0/B in a rather complex way. It was also shown that the solution

Figure 22: Ratio of experimental to theoretical minimum specific energy as a function of y_0/b . The upper and lower plots refer to the lower and upper hysteresis boundaries, respectively.

provided by the one-dimensional approach is not very different from the experimental one, although the extension of the experimental hysteresis region was found to be much smaller than the one predicted by the one-dimensional model.

6 Conclusions

The hysteretic behavior of a free surface steady flow approaching an obstacle was examined. A simple one-dimensional theoretical approach to predict conditions for the occurrence of hydraulic hysteresis and to evaluate the boundaries of the hysteresis region for a supercritical undisturbed approaching flow was proposed.

The theoretical approach was described in detail for three different obstacles in a rectangular channel, namely a sill, a vertical sluice gate, and a vertical circular cylinder.

In all cases, the theoretical boundaries of the hysteresis region were found to be a function of the Froude number of the undisturbed approaching flow, F_0 , and of a geometric parameter characteristic of the type of obstacle.

For the case of flow under a vertical sluice gate, it was also shown that the hysteretic behavior is not characteristic only of supercritical undisturbed approaching

flows but pertains to subcritical undisturbed approaching flows as well provided that the undisturbed Froude number is greater than approximately 0.8.

The reliability of theoretical predictions of hysteresis was tested through comparison with experimental data, either measured by the authors or available in the literature.

It was found that for all investigated obstacles the one-dimensional approach correctly describes the hysteretic behavior of the flow, at least qualitatively. On the contrary, in order to make reliable quantitative predictions, the effects of the physical processes developing at the obstacle had to be evaluated as accurately as possible. In particular, energy losses at the obstacle, effects due to free surface slope and curvature, and contraction phenomena were found to play a crucial role in determining hysteresis boundaries.

Although the above effects can sometimes be suitably evaluated by approximated theoretical expressions, it was shown that in most cases proper experimental investigations are required to correctly predict the amplitude of the hysteresis region and the behavior of hysteresis boundaries as well.

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References

- [1] Abecasis, F.M. & Quintela, A.C., Hysteresis in steady free-surface flow, *Water Power*, **4**, pp. 147–151, 1964.
- [2] Mehrotra, S.C., Hysteresis effect in one- and two-fluid systems, *Proceedings of the V Australian Conference on Hydraulics and Fluid Mechanics*, University of Canterbury, Christchurch, New Zealand, **2**, pp. 452–461, 1974.
- [3] Muskatirovic, D. & Batinic, D., The influence of abrupt change of channel geometry on hydraulic regime characteristics, *Proceedings of the 17th IAHR Congress*, Baden Baden, A, pp. 397–404, 1977.
- [4] Pratt, L.J., A note on nonlinear flow over obstacles, *Geophys. Astrophys. Fluid Dynamics*, **24**, pp. 63–68, 1983.
- [5] Baines, P.G., A unified description of two-layer flow over topography, *J. Fluid Mech.*, **146**, pp. 127–167, 1984.
- [6] Austria, P.M., Catastrophe model for the forced hydraulic jump, *J. Hydraul. Res.*, **25**(3), pp. 269–280, 1987.
- [7] Lawrence, G.A., Steady flow over an obstacle, *J. Hydraul. Eng. ASCE*, **8**, pp. 981–991, 1987.

- [8] Baines, P.G. & Whitehead, J.A., On multiple states in single-layer flows, *Phys. Fluids*, **15**(2), pp. 298–307, 2003.
- [9] Becchi, I., La Barbera, E. & Tetamo, A., Studio sperimentale sul rigurgito provocato da pile di ponte, *Giornale del Genio Civile*, **111**(6-7-8), pp. 277–290, 1973.
- [10] Salandin, P., Indagine sperimentale sulla localizzazione del risalto a mezzo di quinte, *Giornale del Genio Civile*, **131**(1-2-1), pp. 47–71, 1995.
- [11] Defina, A. & Susin, F.M., Hysteretic behavior of the flow under a vertical sluice gate, *Phys. Fluids*, **15**(9), pp. 2541–2548, 2003.
- [12] Henderson, F.M., *Open channel flow*, MacMillan Publishing Co., New York, pp. 67–75, 1966.
- [13] Marchi, E., Effetti locali dovuti alla pendenza e alla curvatura del pelo libero in un restringimento, *Atti del XXIV Convegno di Idraulica e Costruzioni Idrauliche*, Napoli, Italy, **T2**, pp. 35–44, 1994.
- [14] Gentilini, B., Efflusso dalle luci soggiacenti alle paratoie piane inclinate e a settore, *L'Energia Elettrica*, **6**, pp. 361–380, 1941.